Present 2.

9.3 (Canonical transformations)

Let $z \in \mathbb{R}^{2n}$. write $z = \begin{pmatrix} \vec{p} \\ \vec{q} \end{pmatrix}$,

$$\vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} \in \mathbb{R}^n, \quad \vec{q} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} \in \mathbb{R}^n.$$

Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a differentiable function.

Recall we denote the gradient of H by ∇H.

we have (math 2020)

$$\nabla H = \begin{pmatrix} \frac{\partial H}{\partial q_1} \\ \vdots \\ \frac{\partial H}{\partial p_n} \end{pmatrix}$$

Let $E = E_n$ be the $n \times n$ unit matrix.

Let J be the $2n \times 2n$ matrix s.t.

$$J = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$$

we have $J^2 = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} = \left(\begin{array}{c|c} -E^2 & 0 \\ \hline 0 & -E^2 \end{array}\right)$

$= -E_{2n}$ (MATH 1030)

Recall the system $\begin{cases} \dot{q}_1 = \frac{\partial H}{\partial p_1}, & \dot{q}_2 = \frac{\partial H}{\partial p_2} \\ \dot{p}_1 = -\frac{\partial H}{\partial q_1}, & \dot{p}_2 = -\frac{\partial H}{\partial q_2} \end{cases}$

$$\Rightarrow \nabla H = \begin{pmatrix} -\dot{p}_1 \\ -\dot{p}_2 \\ \vdots \\ \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} = \begin{pmatrix} -\dot{p} \\ \dot{q} \end{pmatrix}$$

(note: By MATH2040, we have $\mathbb{R}^n \oplus i\,\mathbb{R}^n \equiv \mathbb{C}^n$)

we have
$$J\dot{z} = J\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix}$$
$$= \left(\begin{array}{c|c} 0 & -E \\ \hline E & 0 \end{array}\right)\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix}$$
$$= \left(\begin{array}{c} -\dot{p}E \\ \hline \dot{q}E \end{array}\right) = \begin{pmatrix} -\dot{p} \\ \dot{q} \end{pmatrix} = \nabla H.$$

So we have got the Hamilton equations can be written as $J\dot{z} = \nabla H$ ...... (9.8)

<u>Def</u>: C^k function (Recall)

Let $\Omega \subset \mathbb{R}^n$ be a open subset.

$f: \Omega \to \mathbb{R}$ be a continuous function.

we say f is a C^k-function if all partial derivatives of order at most k,

$$\partial^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x^\alpha} := \frac{\partial^{|\alpha|} f}{(\partial x^1)^{\alpha_1} \cdots (\partial x^n)^{\alpha_n}}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n \le k$$

exist and continuous on Ω.

we say f is a smooth function if it is a C^k function $\forall k \in \mathbb{N}$.

(C^∞ - function)

Now Define $\Omega \subset \mathbb{R}^m$ be a open sub set.

Let $f = (f_1, f_2, \dots, f_m) : \Omega \to V$ be continuous.

we say f is smooth map if f_i is a smooth map $\forall i \in [\![1, n]\!]$.

<u>Defition</u> (differential of f)

Let $f : \Omega \to V$ be a smooth map. The differential of f assigns to each point $x \in \Omega$ is a linear map $df_x : \mathbb{R}^n \to \mathbb{R}^m$ whose matrix is the Jacobian matrix of f at x, is given by

$$df_x = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix}$$

<u>Defition</u> (Diffemorphism).

Let $\Omega \subset \mathbb{R}^n$ be open subset.

Let $U \subset \mathbb{R}^m$ be open subset.

Let $f : \Omega \to U$ be a smooth map.

f is said to be a diffemorphism if f is bijective and $f^{-1} : U \to \Omega$ is also smooth

Remark: ① clearly if $f : U \to V$ is a diffemorphism, then f^{-1} is also a diffemorphism

② composition of diffeomorphism is also a diffeomorphism

Now we consider a diffeomorphism $w \mapsto z = \varphi(w)$ of $\mathbb{R}^{2n}$, and define K as the pull back of H under this diffeomorphism

i.e. $K(w) = H(\varphi(w))$

we then compute $\nabla K(w)$, we have

$$\nabla K(w) = \left(\frac{\partial K}{\partial w}(w)\right)^t = \left(\frac{\partial H}{\partial z}(z) \cdot \frac{\partial \varphi}{\partial w}(w)\right)^t$$

chain rule (math 2020)

So we have.

$$\nabla K(w) = \left(\frac{\partial K}{\partial w}(w)\right)^t = \left(\frac{\partial H}{\partial z}(z) \cdot \frac{\partial \varphi}{\partial w}(w)\right)^t$$

$$= \left(\frac{\partial \varphi}{\partial w}\right)^t \nabla H$$

$$= \left(\frac{\partial \varphi}{\partial w}\right)^t J \dot{z} \quad (\text{by } 4.8)$$

$$= \left(\frac{\partial \varphi}{\partial w}\right)^t J \frac{\partial \varphi}{\partial w} \dot{w}$$

$\left(\text{since } z = \varphi(w) \Rightarrow \frac{\partial z}{\partial t} = \frac{\partial \varphi}{\partial w} \frac{\partial w}{\partial t} = \frac{\partial \varphi}{\partial w} \dot{w}\right)$

It give a motivation to the following defition.

Def (9.18) canonical transformation.

we say a diffeomorphism φ of open subsets of $\mathbb{R}^{2n}$ is called a canonical transformation If the Jacobian matrix $\phi := \frac{\partial \varphi}{\partial w}$ satisfies.

$$\phi^t J \phi = J \quad \forall w \in \text{domain}(\varphi)$$

(Remark: our computation shows that the canonical transformation are just diffeomorphism that preverse the form of Hamilton equations.)

also note that ϕ is invertiable since

$$|\phi^t J \phi| = |J| \Rightarrow |\phi|^2 = 1 \Leftrightarrow |\phi| \neq 0$$

So we have φ is a local diffeomorphism (MATH 4030)

$|J| \neq 0$ as $J^2 = -E_{2n}$

$$\Rightarrow |J|^2 = 1 \Rightarrow |J| \neq 0$$

Def (9.19) Symplectic.

Any real $2n\times 2n$ matrix ϕ satisfy that

$$\phi^t J \phi = J \text{ is called symplectic.}$$

Example (J is symplectic)

Proof: since we have shown that $J^2 = -E_{2n}$

also we have $J^t = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}^t = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} = -J$.

hence we have $J^t J J = -J^2 \cdot J = E_{2n} J = J$.

Example 2

The 2×2 matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$ is symplectic $\forall \lambda \in \mathbb{R}\setminus\{0\}$

Proof:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}^{-1} = \frac{1}{\lambda^2}\begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\Rightarrow \frac{1}{\lambda^2}\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \lambda \end{pmatrix}$$

$$= \frac{1}{\lambda^2}\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \lambda \end{pmatrix}$$

$$= \frac{1}{\lambda^2}\begin{pmatrix} 0 & -\lambda \\ \frac{1}{\lambda} & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \lambda \end{pmatrix}$$

$$= \frac{1}{\lambda^2}\begin{pmatrix} 0 & -\lambda^2 \\ \frac{1}{\lambda^2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = J.$$

Def differential 1-form (Recall MATH2020)

A differential 1-form in $\mathbb{R}^{2n}$ is a linear combination of the symbols $dx_1, dx_2, \ldots\ldots, dx_{2n}$:

$w = \sum_{i=1}^{2n} w_i dx_i$ with coefficients w_i are function

e.g. the total differential of a smooth function $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a differential 1-form:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

Def: wedge product ($\wedge$)

Let $\wedge$ be a operation s.t.

$$\begin{cases} dx_i \wedge dx_i = 0 & \forall i \in [\![1, 2n]\!] \\ dx_i \wedge dx_j = -dx_j \wedge dx_i & \forall i, j \in [\![1, 2n]\!] \end{cases}$$

and take the usual rules in arithmetic:

i.e. If $w = \sum_{i=1}^{2n} w_i dx_i$, $\gamma = \sum_{i=1}^{2n} \gamma_i dx_i$

then

$$w \wedge \gamma = \left(\sum_{i=1}^{2n} w_i dx_i\right) \wedge \left(\sum_{i=1}^{2n} \gamma_i dx_i\right)$$

$$= \cancel{w_1 dx_1 \wedge \gamma_1 dx_1} + w_2 dx_2 \wedge \gamma_1 dx_1 + \ldots\ldots + w_{2n} dx_{2n} \wedge \gamma_1 dx_1 +$$
$$w_1 dx_1 \wedge \gamma_2 dx_2 + \cancel{w_2 dx_2 \wedge \gamma_2 dx_2} + \ldots + \ldots\ldots + w_{2n} dx_{2n} \wedge \gamma_2 dx_2 + \ldots + \cancel{w_{2n} dx_{2n} \wedge \gamma_{2n} dx_{2n}}$$

$$= \sum_{\substack{i=1,\ldots,2n \\ j=1,\ldots,2n \\ i \neq j}} (w_i \gamma_j - w_j \gamma_i)\, dx_i \wedge dx_j$$

Def differential 2-form (on $\mathbb{R}^{2n}$)

Linear combinations of $dx_i \wedge dx_j$, $i \neq j \in [\![1, 2n]\!]$ are called differential 2-form (on $\mathbb{R}^{2n}$)

$$w = \sum_{i=1}^{2n-1} w_i (dx_i \wedge dx_{i+1}) + w_{2n} (dx_{2n} \wedge dx_1)$$

Similarly, if w is a 1-form and γ is a 2-form, then we can define $w \wedge \gamma$

Note that we insist on the anti-commutativity of wedge product. And we can see that, as $\dim(\mathbb{R}^{2n}) = 2n$, all "linear combinations" are just $f \cdot (dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{2n})$

In summary, we have

0-form : f

1-form = $w_1 dx_1 + w_2 dx_2 + w_3 dx_3 + \cdots + w_{2n} dx_{2n}$

2-form = $\gamma_1 dx_1 \wedge dx_2 + \gamma_2 dx_2 \wedge dx_3 + \gamma_3 dx_3 \wedge dx_4 + \cdots + \gamma_{2n} dx_{2n} \wedge dx_1$

$\vdots$

2n-form : $g(dx_1 \wedge dx_2 \wedge dx_3 \wedge \cdots \wedge dx_{2n})$

(where f, g, w_i, γ_i are function from $\mathbb{R}^{2n} \to \mathbb{R}$).

#

we now want to write the Hamilton equation as an equation in differential forms.

Write $\partial_{q_i}, \partial_{p_i}$ for the unit tangent vector at a given point in $\mathbb{R}^{2n}$ in the coordinate direction of q_i, p_i, respectively. Recall that on a tangent vector

$$\begin{pmatrix} v \\ u \end{pmatrix} = \sum_{i=1}^{n} v_i \partial_{q_i} + \sum_{i=1}^{n} u_i \partial_{p_i}$$

clearly we have

$$dq_j\begin{pmatrix} v \\ u \end{pmatrix} = v_j, \quad dp_j\begin{pmatrix} v \\ u \end{pmatrix} = u_j$$

The wedge product of two 1-forms dp_i and dq_i is the 2 form which evaluates on two vector fields Z, Z' on $\mathbb{R}^{2n}$ as

$$(dp_i \wedge dq_i)(Z, Z') = dp_i(Z)\, dq_i(Z') - dq_i(Z)\, dp_i(Z')$$

<u>Def</u> (9.22) canonical symplectic form.

The canonical symplectic form on $\mathbb{R}^{2n}$ is the differential 2-form

$$\omega := \sum_{i=1}^{n} dp_i \wedge dq_i$$

Lemma (9.23)

The Hamilton equation $J\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix} = \nabla H(q,p)$ is equivalent to $\begin{pmatrix}\dot{q}\\ \dot{p}\end{pmatrix} = X_H(q,p)$,

where the Hamiltonian vector field X_H is defined by

$$\omega(X_H, \cdot) = -dH.$$

Proof: since $dH = \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp$

by defining equation for X_H, we have

$$X_H = \begin{pmatrix} \partial H/\partial p_1 \\ \vdots \\ -\partial H/\partial q_1 \\ \vdots \\ -\partial H/\partial q_n \end{pmatrix} \Rightarrow JX_H = \begin{pmatrix} \partial H/\partial q_1 \\ \vdots \\ \partial H/\partial p \\ \vdots \end{pmatrix} = \nabla H.$$

Prop (9.24)

The autonomous Hamiltonian function H is a constant of motion of the corresponding Hamiltonian system, i.e. the flow lines of X_H are tangent to the level sets of H.

Proof: Skip (trivial follow from the fact that the differential 2-form ω is skew symmetry.

$$dH(X_H) = -\omega(X_H, X_H) = 0$$ (MATH4030)

Prop (9.25)

The transformation $\varphi : (Q, P) \longmapsto (q, p)$ is

canonical $\Longleftrightarrow \varphi^*(dp \wedge dq) = dP \wedge dQ$.

Proof: Let $Z = \begin{pmatrix} X \\ Y \end{pmatrix}$ be a vector field on $\mathbb{R}^{2n}$, where X, Y are $\mathbb{R}^n$ valued. We just use Z' for corresponding notation for a second vector field Z'.

we compute $(dp \wedge dq)(Z, Z')$

$$= YX' - XY'$$
$$= (X^t, Y^t)\begin{pmatrix} -Y' \\ X' \end{pmatrix}$$
$$= Z^t J Z'$$

we still use ϕ as Jacobian of φ, we have

$$\varphi^*(dp \wedge dq)(W, W') = (dp \wedge dq)(\phi W, \phi W')$$
$$= (\phi W)^t J \phi W' = W^t \phi^t J \phi W'$$

so $\varphi^*(dp \wedge dq) = dP \wedge dQ \Leftrightarrow \phi^t J \phi = J$.

Remark 9.26

By result above, canonical transformations are also referred to as symplectomorphism.

9.4 (Equilibrium points and stability)

A point $(q^0, p^0) = (q_1^0, \ldots, q_n^0, p_1^0, \ldots, p_n^0)$ is called an equilibrium point of a given Hamiltonian system.

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \ldots, n$$

If the constant map $t \mapsto (q^0, p^0)$ is a solution of the Hamilton equations.

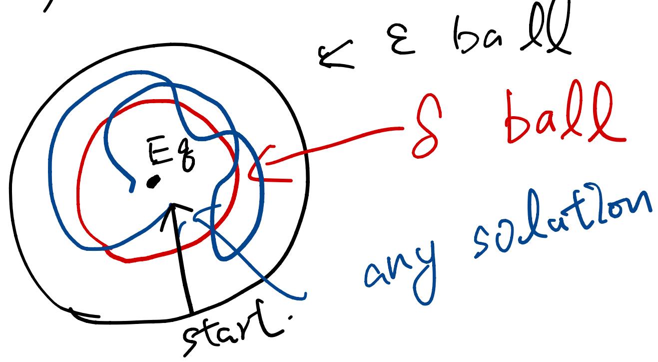
(9.4.1) Lyapunov stability

$$d((q,p),(q^0,p^0)) := \sqrt{\sum_{i=1}^{n} \left((q_i - q_i^0)^2 + (p_i - p_i^0)^2\right)}$$

euclidean norm.

Def 9.28 (Lyapunov stable)

A equilibrium point (q^0, p^0) is called Lyapunov stable if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. If $d((q(0), p(0)), (q^0, p^0)) < \delta$ then the solution t is defined for all $t \in \mathbb{R}_0^+$ and $d((q(t), p(t)), (q^0, p^0)) < \varepsilon \quad \forall t \in \mathbb{R}_0^+$.



Example 9.29

For the Hamiltonian $H(q,p)=\frac{1}{2}(p^2+q^2)$ we have the Hamiltonian equation are

$$\dot{q}=p,\ \dot{p}=-q.$$

The solution are of form

$$q(t)=b\sin(t-a),\ p(t)=b\cos(t-a)$$

$\Rightarrow$ $(0,0)$ is the equilibrium point.

this point is Lyapunov stable.

For $H(q,p)=\frac{1}{2}(p^2-q^2)$

we have the Hamiltonian equation are

$$\begin{cases}\dot{q}=p\\ \dot{p}=q\end{cases}$$

The solution are of the form

$$q(t)=p(t)=be^{t}$$

or

$$q(t)=-p(t)=be^{-t}$$

or

$$\begin{cases}q(t)=b\sinh(t-a)\\ p(t)=b\cosh(t-a)\end{cases}$$

or

$$\begin{cases}q(t)=b\cosh(t-a)\\ p(t)=b\sinh(t-a)\end{cases}$$

$\Rightarrow$ $(0,0)$ is equilibrium point but not Lyapunov stable.

9.4.2 (Linear Stability)

we want to understand the stability in the case of a linear system $\dot{h} = Ah$, where A is a constant $n \times n$ matrix, solution $t \mapsto h(t) \in \mathbb{R}^n$, $t \in \mathbb{R}$. The origin 0 is an equilibrium point of this system.

By ODE, the components of any solution of this linear equation can be written as linear combinations of

① $t^k e^{\lambda t}$ ② $t^k e^{at} \cos bt$ ③ $t^k e^{at} \sin bt$.

where λ can be any real eigenvalue, $a \pm ib$ can be any complex eigenvalue of A. $k \in \mathbb{N}_0$ can take any value from 0 up to $k_{max}-1$, which is defined to be the largest Jordan block for the corresponding eigenvalue being a $k \times k$ matrix

The condition $k_{max} = 1$ for an eigenvalue is equivalent to saying that the geometric multiplicity of this eigen value is equal to it algebraic multiplicity.

As characteristic polynomial is real $\Rightarrow$ if $u \in \mathbb{C}$ is a eigenvalue $\Rightarrow \bar{u}$ is also a eigenvalue.

Prop 9.30

The origin is a stable equilibrium of the linear system $\dot{h} = Ah$ if and only if there are no eigenvalues with positive real part, and $k_{max} = 1$ $\forall$ pure imaginary eigenvalues.

Proof: Suppose there are no eigenvalues with positive real part, and $k_{max} = 1$ for all pure imaginary eigenvalues.

by perious, If l is a solution, then l can be written as linear combination of $t^k e^{\lambda t}, t^k e^{at}\cos bt, t^k e^{at}\sin bt$.

By assumption we have $\lambda < 0$ and $a < 0$, also

$k_{max} = 1 \Rightarrow k = 0$. $\forall$ pure imaginary eigenvalues.

so $t^k e^{\lambda t} \to 0$ as $t \to \infty$

$t^k e^{at}\cos bt < t^k e^{at} \to 0$ as $t \to \infty$

$t^k e^{at}\cos bt < t^k e^{at} \to 0$ as $t \to \infty$

$\Rightarrow$ the origin is a stable equilibrium.

($\Rightarrow$) suppose origin is a stable equilibrium of the linear system $\dot{h} = Ah$

$\Rightarrow$ linear combination of $t^k e^{\lambda t}, t^k e^{at}\cos bt$, $t^k e^{at}\sin bt$ near 0

$\Rightarrow a, \lambda < 0$ (easy!)

If $k_{max} \neq 1 \Rightarrow$ k can be greater than 1.

$\Rightarrow$ t^k is not bounded, contradiction.

#

we now wish to apply this Prop to a given equilibrium point (q^0, p^0) of a Hamiltonian system by linearising the equation near that points.

Let X be a continuous vector-field defined in a neighbourhood $\Omega \subset \mathbb{R}^n$ of some point x^0 with $X(x^0) = 0$.
then $\dot{x} = X(x)$ has an equilibrium point at x^0

If X is differentiable, we can linearising it by $x = x^0 + h$, so we can write $X(x) = Ah + b(h)$, with $A_{ij} = \frac{\partial X_i}{\partial x_j}(x^0)$ is a real $n \times n$ matrix. and b a remainder term satisfying $\lim_{h \to 0} \frac{b(h)}{|h|} = 0$

Def (9.31) Infinitesimally stable.

A equilibrium point x^0 of the dynamical system $\dot{x} = X(x)$ is called infinitesimally stable if the origin is a Lyapunov stable equilibrium of the linearised system $\dot{h} = Ah$.

remark

Infinitesimally stable $\not\Rightarrow$ Lyapunov stable.

(i) If all eigenvalue of A have negative real part, x^0 is even asymptotically stable.
i.e. solutions of $\dot{x} = X(x)$ with initial point sufficiently close to x^0 converge to x^0

(ii) If $\exists$ eigenvalue of A has positive real part
$\Rightarrow$ x^0 is instable. (shown in perious)

---

we now back to the study the stability of an equilibrium point (q^0, p^0) of a Hamiltonian system, where we assume the Hamiltonian function at least C^2. Define ε_i, δ_i by

$$q_i = q_i^0 + \varepsilon_i \ , p_i = p_i^0 + \delta_i \ , i \in [\![1, n]\!]$$

The linear approximation of Hamilton equations at the equilibrium point is then given by

$$\begin{pmatrix} \dot{\varepsilon} \\ \dot{\delta} \end{pmatrix} = \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} \quad \cdots\cdots (9.10),$$

where

$$A_{ij} = \left(\frac{\partial^2 H}{\partial p_i \partial q_i}\right)^0, \ B_{ij} = \left(\frac{\partial^2 H}{\partial p_i \partial p_j}\right)^0, \ C_{ij} = \left(\frac{\partial^2 H}{\partial q_i \partial q_j}\right)^0, \ d_{ij} = \left(\frac{\partial^2 H}{\partial q_i \partial p_j}\right)^0$$

0 means that it evalution at (q^0, p^0).

note that the characteristic roots are λ s.t.

$$\begin{vmatrix} A - \lambda E & B \\ -C & -D - \lambda E \end{vmatrix} \text{ vanishes (math2040)}$$

$$\Leftrightarrow \mathcal{X}(\lambda) := \begin{vmatrix} A - \lambda E & B \\ C & D + \lambda E \end{vmatrix} \text{ vanishes.}$$

Lemma 9.32

The polynomial $\mathcal{X}$ is even

i.e it only contain multiples of $1, \lambda^2, \lambda^4, \ldots, \lambda^{2n}$.

proof: we want to show that $\mathcal{X}(-\lambda) = \mathcal{X}(\lambda)$

Since $D = A^t$, B, C are symmetric matrix and $|M^t| = |M| \ \forall M$ is a $n \times n$ matrix, we have

$$\mathcal{X}(-\lambda) = \begin{vmatrix} A + \lambda E & B \\ C & A^t - \lambda E \end{vmatrix}^t = \begin{vmatrix} A^t + \lambda E & C \\ B & A - \lambda E \end{vmatrix}$$

$$= (-1)^n \begin{vmatrix} C & A^t + \lambda E \\ A - \lambda E & B \end{vmatrix} = \begin{vmatrix} A - \lambda E & B \\ C & A^t + \lambda E \end{vmatrix} = \mathcal{X}(\lambda)$$

Prop 9.33

The equilibrium point (q^0, p^0) is infinitesimally stable if and only if all λ_j are pure imaginary and have $k_{max} = 1$.

($\Rightarrow$) suppose (q^0, p^0) is infinitesimally stable.

$\Rightarrow$ origin is a Lyapunov stable equilibrium. of the linearised system $\dot{h} = Ah$.

by Prop 9.3 $\Rightarrow$ all λ_j with real part ≤ 0 and $k_{max} = 1$. $\forall$ pure imaginary eigenvalue.

by lemma 9.32, the characteristic roots can be written as $\pm\lambda_1, \dots, \pm\lambda_n \in \mathbb{C}$

so if $\exists\, \lambda_j$ with real part < 0

$\Rightarrow \exists\, \lambda_j'$ with real part > 0

$\Rightarrow$ origin is instable equilibrium

$\Rightarrow$ contradiction

then all λ_j with real part $= 0$

$\Rightarrow$ all λ_j are pure imaginary and have $k_{max} = 1$

#

($\Leftarrow$) part is trivial.

Exercise 9.4, 9.5

9.4

Show that any symplectic $2n\times 2n$ matrix ϕ is invertible by expressing the inverse matrix explicitly in terms of ϕ, ϕ^t, J.

we have $\phi^t J \phi = J$ by defition

since we have $J^{-1} = J^t = -J$

$\Rightarrow \quad J^{-1}\phi^t J \phi = I$

$\Rightarrow \quad (J^{-1}\phi^t J)\phi = I$ // $(-J\phi^t J)$

then we have $\phi^{-1} = J^{-1}\phi^t J$. ⋕

(b) show that the symplectic $2n\times 2n$ matrices form a group under matrix multiplication.

clearly $I J I^{-1} = J \Rightarrow I \in Sp(2n)$

suppose $\phi \in Sp(2n)$ i.e. $\phi^t J \phi = J$.

we have $\phi^{-1} = J^{-1}\phi^t J$

$$\Rightarrow (\phi^{-1})^t J \phi^{-1} = J^t \phi (J^{-1})^t J \phi^{-1}$$
$$= J^t \phi (J^t)^{-1} J \phi^{-1}$$
$$= J^t \phi J^2 \phi^{-1}$$
$$= J^t \phi (-E_{2n}) \phi^{-1}$$
$$= -J^t \cdot I$$
$$= J \quad (\text{since } J^{-1} = J^t = -J)$$

$\Rightarrow \phi^{-1} \in Sp(2n)$

If $\phi, K \in Sp(2n)$, we have

$$(\phi K)^t J (\phi K) = K^t \phi^t J \phi K$$
$$= K^t J K = J$$

$\Rightarrow Sp(2n)$ is closed under operation.

the associativity followed by matrix multiplication.

$\Rightarrow Sp(2n)$ is a group

9.5 Write a real $2n \times 2n$ matrix ϕ as a block Matrix
$$\phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A, B, C, D \in M_{n \times n}(\mathbb{R})$$

(a) show that ϕ is symplectic if and only if
$$A^t C = C^t A, \quad B^t D = D^t B, \quad A^t D - C^t B = E_n$$

proof: as ϕ is symplectic, we have

$$\phi^t J \phi = J.$$

$$\Leftrightarrow \begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix} \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} C^t & -A^t \\ D^t & -B^t \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} C^t A - A^t C & C^t B - A^t D \\ D^t A - B^t C & D^t B - B^t D \end{pmatrix} = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} C^t A - A^t C = 0 \\ D^t A - B^t C = E_n \\ D^t B - B^t D = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} C^t A = A^t C \\ A^t D - C^t B = E_n \\ D^t B = B^t D \end{cases} \quad \#$$